Hamiltonians for Safety Technologies

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1 Lagrangians

1.1 One constraint

1.1.1 Result

The problem:

$$\max_{\{x,y\}} f(x,y) \text{ subject to the constraint } g(x,y) = 0 \quad (1)$$

is solved by:

$$\frac{\partial \mathcal{L}}{\partial x} = 0, \quad \frac{\partial \mathcal{L}}{\partial y} = 0, \quad \frac{\partial \mathcal{L}}{\partial \lambda} = 0 \quad (2)$$

Where

$$\mathcal{L} = f + \lambda g. \quad (3)$$

1.1.2 Explanation

Imagine we have a function $f(x,y)$ that we want to maximise subject to the constraint $g(x,y) = 0$. If we move around the $xy$ plane at a maximum of $f$, $f$ cannot be increasing in any direction unless $g$ is also increasing in that direction. This is because if $f$ did increase when we moved around in the $xy$ plane while $g$ remained equal to 0, then we would by definition not be at a maximum of $f$ for the constraint $g = 0$. Therefore, at the maximum, the gradient of $g$ and the gradient of $f$ must point in the same direction:

$$\nabla f = -\lambda \nabla g \Rightarrow \nabla (f + \lambda g) = 0 \quad (4)$$

Where $\lambda$ is constant with respect to $x$ and $y$, and the choice of sign is arbitrary. This condition can be written as:

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \quad (5)$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \quad (6)$$

We can define a Lagrangian, $\mathcal{L}$:

$$\mathcal{L} = f + \lambda g \quad (7)$$

So that, taking the partial derivatives of $\mathcal{L}$ with respect to $x$, $y$, and $\lambda$ and setting them equal to zero, we have:

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0, \quad (8)$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \quad (9)$$
\[ g = 0. \quad (10) \]

The first two equations are the conditions we just found for a maximum of \( f \), and the last equation is the constraint. So, by maximising the Lagrangian with respect to \( x \), \( y \), and \( \lambda \), we can solve the constrained optimisation problem.

### 1.2 Many Constraints

#### 1.2.1 Result

The problem:

\[
\max_x f(x) \text{ subject to the constraints } g(x) = 0 \quad (11)
\]

where \( x = [x_1, x_2, \ldots, x_n] \), is solved by:

\[
\frac{\partial L}{\partial x_i} = 0, \quad (12)
\]

\[
\frac{\partial L}{\partial \lambda_i} = 0, \quad (13)
\]

Where

\[
L = f + \lambda \cdot g, \quad (14)
\]

\( i \) runs from 1 to \( n \), and \( \lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n] \) is constant vector with respect to \( x \).

#### 1.2.2 Explanation

Imagine we have a function \( f(x) \) that we want to maximise subject to the constraint \( g(x) = 0 \) where \( x = [x_1, x_2, \ldots, x_n] \). If we move around in the \( n \)-dimensional \( x \) space at a maximum of \( f \), \( f \) cannot be increasing in any direction unless some linear combination of the constraints \( g \) is also increasing in that direction. This is because if \( f \) did increase when we moved around in the \( x \) space while all the constraints remained equal to 0, then we would by definition not be at a maximum of \( f \) for the constraint \( g = 0 \). Therefore, at the maximum, the gradient of \( g \) and the gradient of \( f \) must point in the same direction:

\[
\nabla f = -\lambda \nabla \cdot g \quad (15)
\]

Where \( \lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n] \) is constant vector with respect to \( x \), and the choice of sign is arbitrary. This condition can be written as:

\[
\frac{\partial f}{\partial x_i} = -\lambda_i \frac{\partial g_i}{\partial x_i} \quad (16)
\]

where \( i \) runs from 1 to \( n \).

We can define a Lagrangian, \( L \):

\[
L = f + \lambda \cdot g \quad (17)
\]
So that, taking the partial derivatives of $\mathcal{L}$ with respect to $x$ and $\lambda$ and setting them equal to zero, we have:

$$\frac{\partial f}{\partial x_i} + \lambda_i \frac{\partial g_i}{\partial x_i} = 0,$$

(18)

$$g = 0$$

(19)

The first $i$ equations are the conditions we just found for a maximum of $f$, and the last equation is the constraint. So, by maximising the Lagrangian with respect to $x$, and $\lambda$, we can solve the constrained optimisation problem.
2 Hamiltonians

2.1 Derivation from Lagrangian

2.1.1 Result

The problem:

\[
\max_x \int_{t_1}^{t_2} I(x) dt \quad \text{subject to the constraint} \quad \dot{x}(x, u, t) = f(x, u, t)
\]  

(20)

where \(x(t) = [x_1(t), x_2(t), ..., x_n(t)]\) are state variables, and \(u(t) = [u_1(t), u_2(t), ..., u_n(t)]\) are control variables has the solution:

\[
\frac{\partial H}{\partial u} = 0
\]

(21)

\[
\frac{\partial H}{\partial x} = -\dot{\lambda}(t)
\]

(22)

Where the Hamiltonian, \(H\) is:

\[
H = I + \lambda(t) \cdot f.
\]

(23)

and \(\lambda(t) = [\lambda_1(t), \lambda_2(t), ..., \lambda_n(t)]\) is a constant vector with respect to \(x\).

2.1.2 Explanation

Imagine we want to optimise the payoff from \(t_1\) to \(t_2\) of some instantaneous payoff function \(I((x(t)), (u(t)), t)\) subject to some instantaneous constraints in the form of a dynamical system \(\dot{x} = f((x(t)), (u(t)), t)\).

\(x = [x_1, x_2, ..., x_n]\) are the state variables and \(u = [u_1, u_2, ..., u_n]\) are the control variables. As before, we can define a Lagrangian for the problem:

\[
\mathcal{L} = \int_{t_1}^{t_2} \mathcal{I} dt + \int_{t_1}^{t_2} \lambda(t) \cdot (f - \dot{x}) dt
\]

(24)

The first term is the function to be optimised, and the second term is the constraint. We can let \(\lambda = \lambda(t)\) so that we are solving for the trajectory of the system from \(t_1\) to \(t_2\).

We can integrate \(\lambda(t) \cdot \dot{x}\) by parts:

\[
\int_{t_1}^{t_2} \lambda(t) \cdot \dot{x} dt = [\lambda(t) \cdot x]_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{\lambda}(t) \cdot x dt.
\]

(25)

Substituting this back into the Lagrangian, we have:

\[
\mathcal{L} = \int_{t_1}^{t_2} (I + \lambda(t) \cdot f + \dot{\lambda}(t) \cdot x) dt + [\lambda(t) \cdot x]_{t_1}^{t_2}.
\]

(26)
If we are at an optimum of $L$ then varying $x$ and $u$ by some amounts $dx$ and $du$ should leave $L$ unchanged i.e. $dL = 0$.

$$dL = \int_{t_1}^{t_2} (\frac{\partial I}{\partial x} \cdot dx + \frac{\partial I}{\partial u} \cdot du + \lambda(t) \cdot (\frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial u} \cdot du) + \dot{\lambda}(t) \cdot dx)dt + [\lambda(t) \cdot dx]_{t_1}^{t_2}$$

(27)

Collecting coefficients of $dx$ and $du$ from inside the integral and setting these equal to zero:

$$\frac{\partial I}{\partial x} + \lambda(t) \cdot \frac{\partial f}{\partial x} + \dot{\lambda}(t) = 0 \quad \text{and} \quad (28)$$

$$\frac{\partial I}{\partial u} + \lambda(t) \cdot \frac{\partial f}{\partial u} = 0 \quad \text{(29)}$$

These two conditions, combined with the boundary constraints at $t_1$ and $t_2$, give the optimal trajectory for the system.

To capture these conditions more conveniently, we can define a Hamiltonian $\mathcal{H}$:

$$\mathcal{H} = I + \lambda(t) \cdot f.$$  \hspace{1cm} (30)

The conditions (28) and (29) are met when:

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \quad \text{(31)}$$

$$\frac{\partial \mathcal{H}}{\partial x} = -\dot{\lambda}(t) \quad \text{(32)}$$

So these are the conditions for a constrained optimum.

### 2.2 Economic Intuition

So far we’ve motivated the Hamiltonian without any economic intuition, but it does have a useful economic interpretation. Consider a modified Hamiltonian $\mathcal{H}' = \mathcal{H} + \lambda(t)x$:

$$\mathcal{H}' = I + \lambda \cdot \dot{x} + \dot{\lambda}(t)x$$  \hspace{1cm} (33)

This modified Hamiltonian is the function we’re optimising (in an unconstrained way). The three terms in this Hamiltonian capture the three sources of value in the system. The first is the payoff function, which is obvious. The second term is the marginal value of state variables. We can see this by noticing that the multiplier $\lambda_i$ works as the shadow price of the $i$th state variable. Then the second term is the rate of change of the state variables $x$ multiplied by the shadow prices of those state variables, $\lambda$. The third term is the change in value of current state variables - the change in price of each variable, $\lambda$ multiplied by the amount of each variable, $x$. 
3 Applications

We can use Hamiltonians to maximise the total payoff of some instantaneous payoff function between a time $t_1$ and $t_2$ subject to constraints in the form of a dynamical system. The instantaneous payoff function can be the (time-discounted) utility of a representative agent and the constraints can be production functions.

3.1 Boundary conditions

The specific problem will often provide boundary conditions. For example, at $t_1$, there may be a given amount of capital stock or consumption technology. There may also be boundary conditions at $t_2$ - for example, given that we are optimising some function only up to $t_2$ and not beyond, we may require that the rate of saving at $t_2$ is zero (because there is no value in saving when there is no time for return on investment). If the time horizon is infinite such that $t_2 \to \infty$, the transversality condition will take the form of a limit, but the economic intuition remains the same - there is no value in investing in capital, technology, or safety is there is no time to reap the rewards.

3.2 Present and current value

Imagine we want to optimise the integral of some exponentially discounted utility $e^{-\rho t} u(c(t))$ subject to some constraints $\dot{x} = f(x)$. The Hamiltonian is:

$$H = e^{-\rho t} u(c(t)) + \lambda \cdot f$$  

(34)

This is the present value Hamiltonian. We can define the current value Hamiltonian $H_{CV}$ by letting $\mu = e^{\rho t} \lambda$ such that:

$$H_{CV} = u(c(t)) + \mu \cdot f$$  

(35)

The conditions on the present value Hamiltonian were:

$$\frac{\partial H}{\partial u} = 0, \quad \frac{\partial H}{\partial x} = -\dot{\lambda}.\tag{36}$$

So the conditions on the current value Hamiltonian are:

$$\frac{\partial H_{CV}}{\partial u} = 0, \quad \frac{\partial H_{CV}}{\partial x} = -\dot{\mu} + \rho \mu.\tag{37}$$

3.3 Example 1: Personal Saving

Imagine we want to find the path of an agent’s wealth, $a$ (the state variable) that maximises their discounted utility $u$ over a period $t_1$ to $t_2$. The function to be maximised is:

$$\int_{t_1}^{t_2} e^{-\rho t} u(c(t))dt.$$

(38)
And the constraint is the budget constraint with savings:

\[ \dot{a}(t) = (R(t) - 1)a(t) + w(t) - c(t). \]  

(39)

Where \( R \) is the interest rate, and \( w \) is the wage. The agent controls how much of their wage to consume or save so \( c \) is the control variable. The Hamiltonian is:

\[ \mathcal{H} = e^{-\rho t} u(c(t)) + \lambda [(R(t) - 1)a(t) + w(t) - c(t)] \]  

(40)

To find the optimal path we need to solve:

\[ \frac{\partial \mathcal{H}}{\partial c} = 0, \]  

(41)

\[ \frac{\partial \mathcal{H}}{\partial a} = -\dot{\lambda}(t), \text{ and} \]  

(42)

\[ \dot{a}(t_2) = 0 \text{ (marginal saving at } t_2 \text{ has no value)} \]  

(43)

3.4 Example 2: Safety Technologies

Consider an economy with a consumption sector producing consumption good \( C \) and a safety sector producing safety good \( H \). Each person in the economy works in either the consumption sector or the safety sector. People employed in a given sector can be workers or scientists. Workers produce goods for their respective sector. Scientists produce technologies for their respective sectors. The control variables for this system are the fraction of scientists that are working in the consumption sector \( s(t) \), the fraction of workers that are working in the consumption sector \( l(t) \), and the fraction of people that are employed as scientists \( \sigma(t) \). Consumption good production per person is given by

\[ c(t) = (1 - \sigma(t))l(t)A(t)^\alpha \]  

(44)

where \((1 - \sigma(t))l(t)\) is the fraction of the population working on producing consumption goods, and \(A(t)\) is consumption technology. The production function for technologies is:

\[ \dot{A}(t) = S_a(t)A(t)^\phi \]  

(45)

where \( S_a \) is the number of scientists researching new consumption technologies, and \( \lambda \) and \( \phi \) are constants. There is an analogous production function for safety goods:

\[ h(t) = (1 - \sigma(t))(1 - l(t))B(t)^\alpha \]  

(46)

where \( h \) is the amount of safety goods per person, \((1 - \sigma(t))(1 - l(t))\) is the fraction of people working on producing safety goods, and \(B\) are safety technologies. The production function for safety technologies is:

\[ \dot{B}(t) = S_b(t)B(t)^\phi \]  

(47)
where $S_b$ is the number of scientists researching new safety technologies. These two dynamical systems are two of our constraints, the third comes in the form of a probability of catastrophe:

$$M(t) = -\delta c^e h^{-\beta} M(t)$$  \(48\)

where $M(t)$ is the probability that there is no catastrophe before time $t$, $\epsilon$ is the potency of consumption goods at increasing catastrophic risk, $\beta$ is the potency of safety goods at decreasing catastrophic risk, and $\delta$ is a constant. So, as well as utility being discounted exponentially at $\rho$, utility at $t$ will be discounted because of empirical uncertainty about survival to time $t$. The function to be maximised is then:

$$\int_t^\infty M(t)e^{-\rho t}u(c(t))dt.$$  \(49\)

And our three constraints are:

$$\dot{A}(t) = S_a(t)^\lambda A(t)^\phi,$$  \(50\)

$$\dot{B}(t) = S_b(t)^\lambda B(t)^\phi,$$  \(51\)

$$\dot{M}(t) = -\delta c^e h^{-\beta} M(t).$$  \(52\)

The Hamiltonian is then:

$$H = M(t)e^{-\rho t}u(c(t))+\lambda_A(t)[S_a(t)^\lambda A(t)^\phi]+\lambda_B(t)[S_b(t)^\lambda B(t)^\phi]+\lambda_M(t)[-\delta c^e h^{-\beta} M(t)]$$  \(53\)

We can redefine the Hamiltonian in terms of current value using the substitution $\mu_i = e^{\rho t}A_i$:

$$H_{CV} = M(t)e^{\rho t}u(c(t))+\mu_A(t)[S_a(t)^\lambda A(t)^\phi]+\mu_B(t)[S_b(t)^\lambda B(t)^\phi]+\mu_M(t)[-\delta c^e h^{-\beta} M(t)]$$  \(54\)

And we can write the aggregate variables $S_a$ and $S_b$ in per capita form using these substitutions (which follow directly from the setup):

$$S_a = \sigma sN, S_b = \sigma (1-s)N$$  \(55\)

Where $N$ is total population and is governed by the equation:

$$\dot{N} = nN$$  \(56\)

where $n$ is a constant. Population growth is exogenous to the model; it is neither a control nor state variable. We can rewrite the Hamiltonian one last time as an explicit function of our control variables $s$, $l$, and $\sigma$ and our state variables $A$, $B$, and $M$:

$$H_{CV} = M(t)e^{\rho t}u(c(t))+\mu_A(t)[(\sigma sN)^\lambda A(t)^\phi]+\mu_B(t)[(\sigma (1-s)N)^\lambda B(t)^\phi]+\mu_M(t)[-\delta c^e h^{-\beta} M(t)]$$  \(57\)
To find the path that optimises \( u(c(t)) \) subject to our constraints we need to solve:

\[
\frac{\partial H_{CV}}{\partial s} = 0, \tag{58}
\]

\[
\frac{\partial H_{CV}}{\partial t} = 0, \tag{59}
\]

\[
\frac{\partial H_{CV}}{\partial \sigma} = 0, \tag{60}
\]

\[
\frac{\partial H_{CV}}{\partial A} = -\dot{\mu}_A + \rho \mu_A, \tag{61}
\]

\[
\frac{\partial H_{CV}}{\partial B} = -\dot{\mu}_B + \rho \mu_B, \tag{62}
\]

\[
\frac{\partial H_{CV}}{\partial M} = -\dot{\mu}_M + \rho \mu_M \tag{63}
\]

Subject to the boundary conditions:

\[
\lim_{t \to \infty} (e^{-\rho t} \mu_A A(t)) = 0, \tag{64}
\]

\[
\lim_{t \to \infty} (e^{-\rho t} \mu_B B(t)) = 0, \tag{65}
\]

\[
\lim_{t \to \infty} (e^{-\rho t} \mu_M M(t)) = 0 \tag{66}
\]

which is just to say that the values of marginal consumption technologies, safety technologies, and extra survival probability go to zero as \( t \to \infty \).